International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 6, Issue 1, Dec – Jan 2017; 55-64

International Academy of Science, Engineering and Technology
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BUSY PERIOD ANALYSIS OF A MARKOVIAN FEEDBACK QUEUEING MODEL WITH SERVERS HAVING UNEQUAL SERVICE RATE

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ABSTRACT

This paper studies a feedback queueing system with two parallel servers having different service rates. Arrivals follow Poisson distribution. Service times for both the servers are exponentially distributed. Busy period distribution for this system is obtained using generating function technique. A few special cases of interest are also derived. Results are illustrated with numerical examples and compared graphically.

KEYWORDS: Busy Period, Feedback, Numerical Solution, Probability, Queueing, Server

INTRODUCTION

The research work in queueing theory led to many extensions in basic queueing theory due to its significance in fields like telecommunication, aviation, manufacturing and production, transportation and many more. A number of authors have studied systems with two servers in parallel. Morse [9] considered two independent branches of service facility, one having the service rate $2(1-\sigma)\mu$, where $\sigma \in \left(0, \frac{1}{2}\right)$. Arrivals join a common waiting line, unit at the head of the waiting line enters one or the other branch of service facility with relative frequencies σ and $1-\sigma$. He obtained the steady state solutions for the cases when no queue is allowed before the service facility.

Saaty [11] studied a continuous time first come first served queueing problem with two parallel servers each having a different service rate. He obtained the steady state probabilities for the number of units in the system/queue. Gumbel [6] considered a more general queueing problem having a finite number of servers, each with a different service rate. He also obtained the steady state probabilities for the number of units in the system. There is now a growing interest in the analysis of queueing systems with feedback. In feedback queueing systems, the customers get additional service if they are not satisfied from their previous service. Many researchers paid their attention towards this direction. Tackas, L.[14] , Thangaraj & Vanitha[13], Shardha & Garg[12] and Kumari[8] studied a number of queueing systems with feedback. Garg and Singla [4] studied a queueing model with two service channels having different service rates. The arriving units are also given the option of rejoining the system with a definite probability after being served once. They obtained transient probabilities and steady-state solution for the model.

Busy period analysis plays a vital role in the study of queueing problems for forecasting the behavior of queueing systems. Performance analysis of busy period is completely dependent on the busy period distribution of the queueing model. In particular, the busy period analysis is important from servers' point of view and is also helpful in the efficient planning of the system resources. The distribution of the busy period lengths has been studied by a number of investigators, among them are Kendall [7], Pollaczek[10], and Kumari[8] .

The present paper is concerned with the busy periods (the periods during which the server is continuously busy) in a feedback queueing system with two servers having different service rates when no restrictions are placed on the queue length. Laplace transformation of probability generating function for busy period of the system is obtained. Results are illustrated with numerical examples and are compared graphically.

The practical situation which corresponds to the above situation can be that of two machines of different service rates engaged in repair work. After first repair the component is checked, if it is not found satisfactory then it is again sent in the queue for second repair otherwise it leaves the system. However after second service the component leaves the system definitely. The Manager can know distribution of busy period duration for the repairshop. *The feedback queueing system investigated in this paper is described by the following assumptions:*

(i) Arrivals are Poisson with parameter λ.

(ii) Service times are exponentially distributed with parameters μ_1 and μ_2 for the first and second channels respectively.

(iii) When both the channels are empty, an arriving unit joins first channel with probability a_1 and second channel with probability a_2, so that $a_1+a_2=1$.

(iv) After current departure, the next customer will depart the service channel for the first time with probability c_1 and for the second time with probability c_2, so that $c_1 + c_2 = 1$.

(v) The probability of rejoining the system is p and that of leaving the system is q for the units getting service first time, so that p+q =1. However the units will have to leave the system definitely after getting service for the second time.

(vi) Units are taken for service in their order of arrival.

(vii) The stochastic processes involved, viz

a. arrival of units

b. departure of units, are statistically independent.

Definitions

$P_n^{(k)}(t)$ = Probability that there are n units in the system at time t and the next unit is to depart for the first time or second time according as k=0 or 1. $n \geq 0$

$P_n(t)$ = Probability that there are n units in the system at time t. $n \geq 0$

$$P_n(t) = P_n^{(0)}(t) + P_n^{(1)}(t), n \geq 0 \tag{2.1}$$

$P_1^{(k)}(1,0,t)$=Probability that the unit is in the first channel at time t and the unit is to depart for the first time or second time according as k=0 or 1.

$P_1^{(k)}(0,1,t)$= Probability that the unit is in the second channel at time t and the unit is to depart for the first time or second time according as k=0 or 1.

$$P_1(1,0,t) = P_1^{(0)}(1,0,t) + P_1^{(1)}(1,0,t)$$

$$P_1(0,1,t) = P_1^{(0)}(0,1,t) + P_1^{(1)}(0,1,t)$$

$$P_1^{(k)}(t) = P_1^{(k)}(1,0,t) + P_1^{(k)}(0,1,t), k = 0,1$$

$$P_1(t) = P_1^{(0)}(t) + P_1^{(1)}(t)$$

$P_1(t) = P_1(1,0,t) + P_1(0,1,t)$ Initially, no unit is present in the system i.e. $P_0^{(0)}(0) = 1$.

BUSY PERIOD DISTRIBUTION

The probability density function for the busy period distribution is given by $\frac{d}{dt}P_0^{(0)}(t)$ and is obtained using the following set of difference -differential equations

$$\frac{d}{dt}P_0^{(0)}(t) = \mu_1\left\{qP_1^{(0)}(1,0,t) + P_1^{(1)}(1,0,t)\right\} + \mu_2\left\{qP_1^{(0)}(0,1,t) + P_1^{(1)}(0,1,t)\right\} \tag{2.2}$$

$$\frac{d}{dt}P_1^{(0)}(1,0,t) = -(\lambda + \mu_1)P_1^{(0)}(1,0,t) + \mu_2 c_1\left\{qP_2^{(0)}(t) + P_2^{(1)}(t)\right\} \tag{2.3}$$

$$\begin{aligned}\frac{d}{dt}P_1^{(1)}(1,0,t) = {} & -(\lambda + \mu_1)P_1^{(1)}(1,0,t) + \mu_2 c_2\left\{qP_2^{(0)}(t) + P_2^{(1)}(t)\right\} \\ & + \mu_1 pa_1 P_1^{(0)}(1,0,t) + \mu_2 pa_1 P_1^{(0)}(0,1,t)\end{aligned} \tag{2.4}$$

$$\frac{d}{dt}P_1^{(0)}(0,1,t) = -(\lambda + \mu_2)P_1^{(0)}(0,1,t) + \mu_1 c_1\left\{qP_2^{(0)}(t) + P_2^{(1)}(t)\right\} \tag{2.5}$$

$$\begin{aligned}\frac{d}{dt}P_1^{(1)}(0,1,t) = {} & -(\lambda + \mu_2)P_1^{(1)}(0,1,t) + \mu_1 c_2\left\{qP_2^{(0)}(t) + P_2^{(1)}(t)\right\} \\ & + \mu_1 pa_2 P_1^{(0)}(1,0,t) + \mu_2 pa_2 P_1^{(0)}(0,1,t)\end{aligned} \tag{2.6}$$

$$\begin{aligned}\frac{d}{dt}P_n^{(0)}(t) = {} & -(\lambda + \mu_1 + \mu_2)P_n^{(0)}(t) + \lambda P_{n-1}^{(0)}(t) \\ & + (\mu_1 + \mu_2)c_1\left\{qP_{n+1}^{(0)}(t) + P_{n+1}^{(1)}(t)\right\} + (\mu_1 + \mu_2)c_1 pP_n^{(0)}(t)\end{aligned} \quad , n \geq 2 \tag{2.7}$$

$$\begin{aligned}\frac{d}{dt}P_n^{(1)}(t) = {} & -(\lambda + \mu_1 + \mu_2)P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) \\ & + (\mu_1 + \mu_2)c_2\left\{qP_{n+1}^{(0)}(t) + P_{n+1}^{(1)}(t)\right\} + (\mu_1 + \mu_2)c_2 pP_n^{(0)}(t)\end{aligned} \quad , n \geq 2 \tag{2.8}$$

with initial conditions $P_1^{(0)}(1,0,0) = a_1$ and $P_1^{(0)}(0,1,0) = a_2$

Taking the Laplace Transformation $\overline{P}_n(s)=\int_0^\infty e^{-st}P_n(t)dt$ **Re** $s>0$ **of (2.2) - (2.8) and dividing by** $(\mu_1+\mu_2)$

$$\left\{\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_0^{(0)}(s)=r_1\left\{q\overline{P}_1^{(0)}(1,0,s)+\overline{P}_1^{(1)}(1,0,s)\right\}+r_2\left\{q\overline{P}_1^{(0)}(0,1,s)+\overline{P}_1^{(1)}(0,1,s)\right\} \tag{2.9}$$

$$\left\{\rho+r_1+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_1^{(0)}(1,0,s)=\frac{a_1}{(\mu_1+\mu_2)}+r_2c_1\left\{q\overline{P}_2^{(0)}(s)+\overline{P}_2^{(1)}(s)\right\} \tag{2.10}$$

$$\left\{\rho+r_1+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_1^{(1)}(1,0,s)=r_2c_2\left\{q\overline{P}_2^{(0)}(s)+\overline{P}_2^{(1)}(s)\right\}+r_1pa_1\overline{P}_1^{(0)}(1,0,s)+r_2pa_1\overline{P}_1^{(0)}(0,1,s) \tag{2.11}$$

$$\left\{\rho+r_2+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_1^{(0)}(0,1,s)=\frac{a_2}{(\mu_1+\mu_2)}+r_1c_1\left\{q\overline{P}_2^{(0)}(s)+\overline{P}_2^{(1)}(s)\right\} \tag{2.12}$$

$$\left\{\rho+r_2+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_1^{(1)}(0,1,s)=r_1c_2\left\{q\overline{P}_2^{(0)}(s)+\overline{P}_2^{(1)}(s)\right\}+r_1pa_2\overline{P}_1^{(0)}(1,0,s)+r_2pa_2\overline{P}_1^{(0)}(0,1,s) \tag{2.13}$$

$$\left\{\rho+1+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_n^{(0)}(s)=\rho\overline{P}_{n-1}^{(0)}(s)+c_1\left\{q\overline{P}_{n+1}^{(0)}(s)+\overline{P}_{n+1}^{(1)}(s)\right\}+c_1p\overline{P}_n^{(0)}(s)\ ,n\ge 2 \tag{2.14}$$

$$\left\{\rho+1+\frac{s}{(\mu_1+\mu_2)}\right\}\overline{P}_n^{(1)}(s)=\rho\overline{P}_{n-1}^{(1)}(s)+c_2\left\{q\overline{P}_{n+1}^{(0)}(s)+\overline{P}_{n+1}^{(1)}(s)\right\}+c_2p\overline{P}_n^{(0)}(s)\ ,n\ge 2 \tag{2.15}$$

where $\rho=\lambda/(\mu_1+\mu_2)$, $r_1=\mu_1/(\mu_1+\mu_2)$, $r_2=\mu_2/(\mu_1+\mu_2)$;

Definitions

$$G^{(0)}(z,t)=\sum_{n=1}^{\infty}P_n^{(0)}(t)z^n \qquad \overline{G}^{(0)}(z,s)=\int_0^\infty e^{-st}G^{(0)}(z,t)\,dt$$

$$G^{(1)}(z,t)=\sum_{n=1}^{\infty}P_n^{(1)}(t)\,z^n \qquad \overline{G}^{(1)}(z,s)=\int_0^\infty e^{-st}G^{(1)}(z,t)\,dt$$

$$G(z,t) = G^{(0)}(z,t) + G^{(1)}(z,t) \qquad \bar{G}(z,s) = \int_0^\infty e^{-st} G(z,t)\,dt \quad \text{with } |z| \leq 1$$

Laplace Transformation of Probability Generating Function for the Busy Period Distribution

$$\bar{G}(z,s) = \frac{\left(\frac{z^2}{\mu_1+\mu_2}\right)B(z) - zA(z)\left\{(q+pz)\bar{P}_1^{(0)}(s)+\bar{P}_1^{(1)}(s)\right\} + z^2\begin{bmatrix} pE(z)\left\{r_1\bar{P}_1^{(0)}(1,0,s)+r_2\bar{P}_1^{(0)}(0,1,s)\right\} \\ +B(z)\left\{r_2\bar{P}_1^{(0)}(1,0,s)+r_1\bar{P}_1^{(0)}(0,1,s)\right\} \\ +E(z)\left\{r_2\bar{P}_1^{(1)}(1,0,s)+r_1\bar{P}_1^{(1)}(0,1,s)\right\}\end{bmatrix}}{\{A(z)-c_2\}\{A(z)-c_1*(q+pz)\}-c_1c_2*(q+pz)}$$

$$\rho = \lambda/(\mu_1+\mu_2)\ ;\ |z| \leq 1 \tag{2.16}$$

where $A(z) = \left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{(\mu_1+\mu_2)}\right)z\right\}$

$$B(z) = \{A(z) - c_2 p * (1-z)\}$$

$$E(z) = \{A(z) + c_1 p * (1-z)$$

Let $D = K_1(z) * K_2(z) - c_1c_2 * (q+pz)$

where $K_1(z) = \left[\left\{-\rho z^2 + \left(\rho+1+\frac{s}{(\mu_1+\mu_2)}\right)z\right\} - c_2\right]$, $K_2(z) = \left[\left\{-\rho z^2 + \left(\rho+1+\frac{s}{(\mu_1+\mu_2)}\right)z\right\} - c_1 * (q+pz)\right]$

Obviously $K_1(z)$ and $K_2(z)$ have two zeroes inside the unit circle.

Let $f(z) = K_1(z) * K_2(z)$ and $g(z) = c_1c_2 * (q+pz)$

$|f(z)| = |K_1(z)| * |K_2(z)|$

$= \left|\left[\left\{-\rho z^2 + \left(\rho+1+\frac{s}{(\mu_1+\mu_2)}\right)z\right\} - c_2\right]\right| * \left|\left[\left\{-\rho z^2 + \left(\rho+1+\frac{s}{(\mu_1+\mu_2)}\right)z\right\} - c_1 * (q+pz)\right]\right|$

$\geq \{\zeta + c_1\}\{\zeta + c_2\}$ for $\frac{s}{(\mu_1+\mu_2)} = \zeta + i\eta$, $|z| = 1 > c_1c_2 \geq |g(z)|$

Hence $|f(z)| > |g(z)|$ on $|z| = 1$

Since all the conditions of Rouche's Theorem are satisfied, so the denominator D in (2.16) has two zeroes inside the unit circle. Let these zeroes be $z_m (m = 0, 1)$. Numerator must vanish for these two zeroes since $\bar{G}(z,s)$ is an analytical function of z. These two equations along with equations (2.10), (2.11), (2.12) and (2.13) will determine the six unknowns $\bar{P}_1^{(0)}(1,0,s)$, $\bar{P}_1^{(0)}(0,1,s)$, $\bar{P}_1^{(1)}(1,0,s)$, $\bar{P}_1^{(1)}(0,1,s)$, $\bar{P}_2^{(0)}(s)$ and $\bar{P}_2^{(1)}(s)$. Hence the generating function $\bar{G}(z,s)$ is completely known. Using Laplace Inverse of $\bar{P}_1^{(0)}(1,0,s)$, $\bar{P}_1^{(0)}(0,1,s)$, $\bar{P}_1^{(1)}(1,0,s)$ and $\bar{P}_1^{(1)}(0,1,s)$ in the equation (2.2), $\frac{d}{dt}P_0^{(0)}(t)$ can be obtained which in turn will give us probability density function of busy period distribution for the model.

SPECIAL CASES

(i) When there is no feedback

Putting $q = 1, p = 0, c_1 = 1, c_2 = 0,\ \bar{P}_1^{(0)}(s) = \bar{P}_1(s), \bar{P}_1^{(1)}(s) = 0,$

$$\overline{G}(z,s) = \frac{z^2\left\{r_2\overline{P}_1(1,0,s) + r_1\overline{P}_1(0,1,s) + \frac{1}{\mu_1+\mu_2}\right\} - z\overline{P}_1(s)}{\left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{(\mu_1+\mu_2)}\right)z\right\} - 1}$$

$\rho = \lambda/(\mu_1 + \mu_2)$; $|z| \le 1$ (2.17)

(ii) When there is only one server i.e. M/M/1 system with feedback

Putting $\mu_1 = \mu, \mu_2 = 0, r_1 = 1, r_2 = 0$,$\overline{P}_1^{(0)}(1,0,s) = \overline{P}_1^{(0)}(s)$, $\overline{P}_1^{(0)}(0,1,s) = 0$,
$\overline{P}_1^{(1)}(1,0,s) = \overline{P}_1^{(1)}(s)$ and $\overline{P}_1^{(1)}(0,1,s) = 0$ in equation (2.16), we get

$$\overline{G}(z,s) = \frac{(z^2/\mu)\{A(z) - c_2 p(1-z)\} - z\left\{\{qA(z) - c_1 p^2 z(1-z)\}\overline{P}_1^{(0)}(s) + A(z)\overline{P}_1^{(1)}(s)\right\}}{\left\{\left(A(z) - c_1(q+pz)\right)\left(A(z) - c_2\right)\right\} - (q+pz)c_1 c_2}$$

$\rho = \lambda/\mu$; $|z| \le 1$ (2.18)

where $A(z) = \left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{\mu}\right)z\right\}$ (iii) **When there is no feedback and there is only one server**

Putting $\mu_1 = \mu, \mu_2 = 0, r_1 = 1, r_2 = 0$, $\overline{P}_1(1,0,s) = \overline{P}_1(s), \overline{P}_1(0,1,s) = 0$ in equation (2.17), we get

$$\overline{G}(z,s) = \frac{\frac{z^2}{\mu} - z\overline{P}_1(s)}{\left\{-\rho z^2 + \left(\rho + 1 + \frac{s}{(\mu_1+\mu_2)}\right)z\right\} - 1} \quad \rho = \lambda/\mu \text{ ; } |z| \le 1 \qquad (2.19)$$

This coincides with Laplace transformation of busy period duration generating function of M/M/1 model.

NUMERICAL AND GRAPHICAL SOLUTION

Numerical solution of equations [(2.2)-(2.8)] is obtained using MATLAB programming and results are presented graphically. In Figure 1, the distribution function and density function for the duration of a busy period is plotted for the case $\rho = 0.3$, $C_1 = 0.6$, $q = 0.75$, $r_1 = 0.6, a_1 = 0.7$.

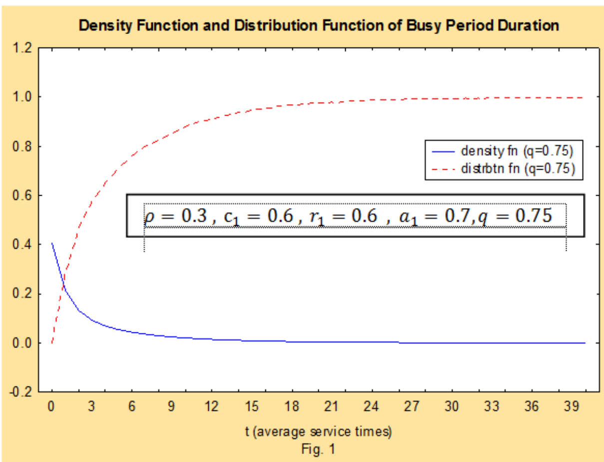

Figure 1

Density function of busy period duration for different values of ρ keeping other parameters constant is plotted in Figure 2. The set of values for ρ is $\{0.2, 0.4, 0.6, 0.8\}$. The other parameters were fixed at $c_1 = 0.6$, $q = 0.75$, $r_1 = 0.6$, $a_1 = 0.7$. Distribution function for busy period duration is also compared for different values of ρ (keeping other parameters constant) through Figure 2.

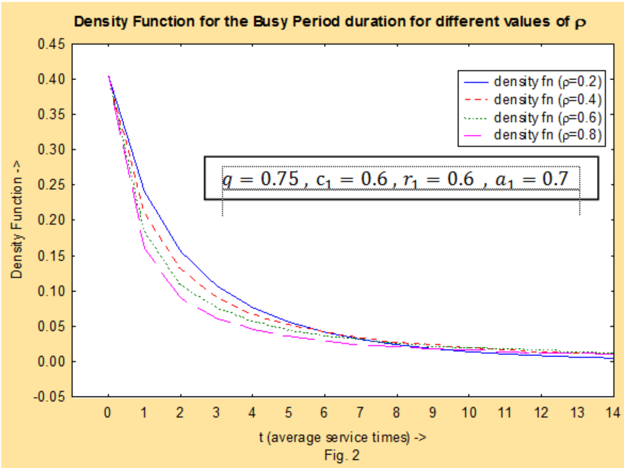

Figure 2

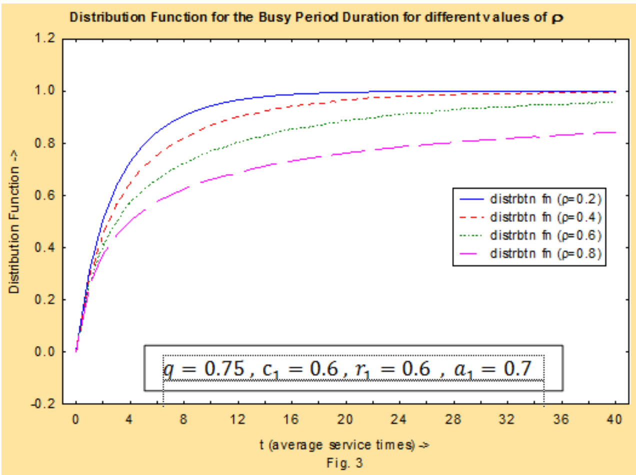

Figure 3

Behavior of the busy period duration density function and distribution function with changing probability q (probability of leaving the system after getting first service) is shown in Figure 4 and Figure 5.

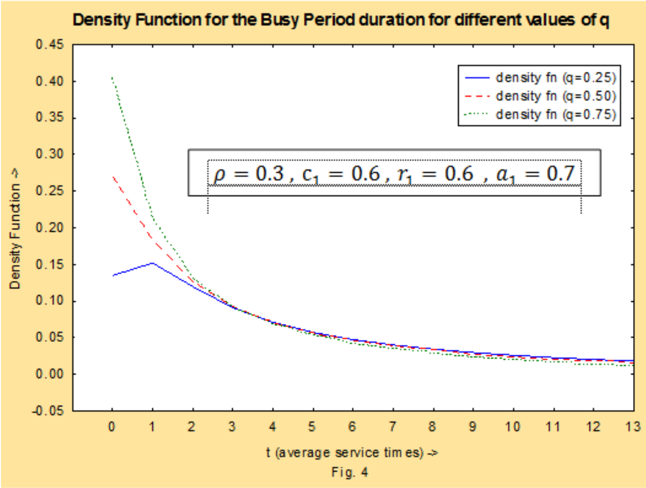

Figure 4

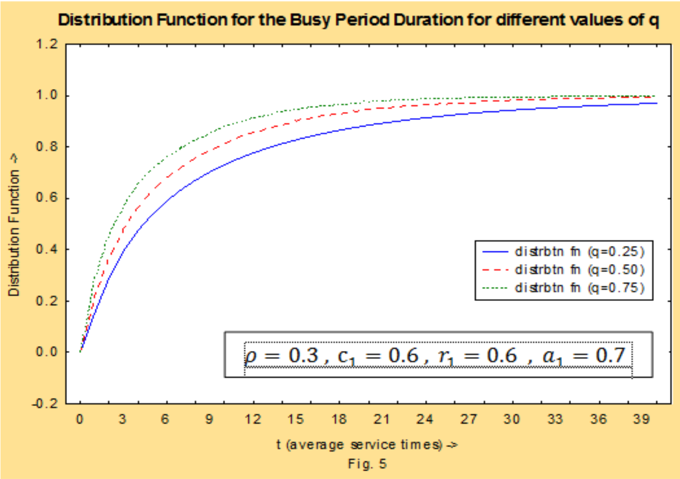

Figure 5

Density function and distribution function of busy period duration when all the servers are busy are compared with those of total busy period duration through Figure 6 and Figure 7.

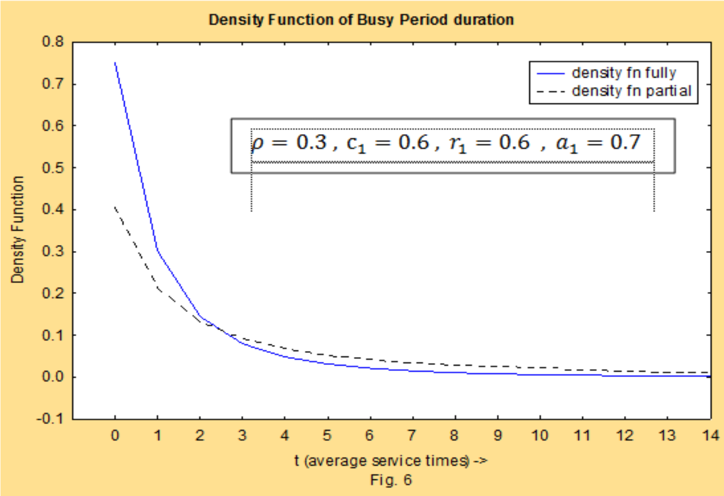

Figure 6

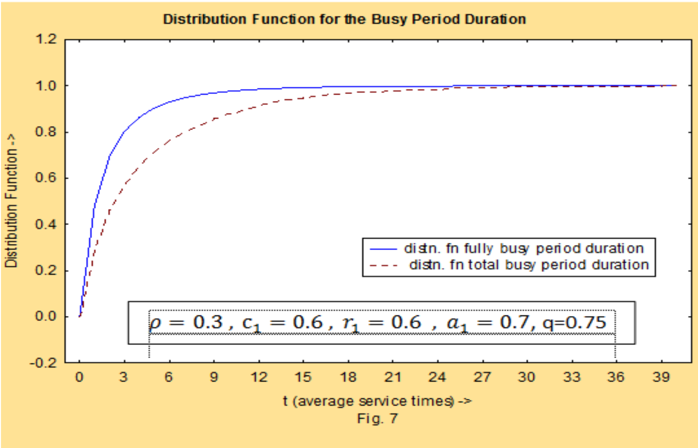

Figure 7

ACKNOWLEDGEMENTS

Author is thankful to Dr. D.K. Garg, Head, Department of Statistics, Punjabi University Patiala for providing necessary facilities during the preparation of the research paper.

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